

CHAPTER 1

The first part of the book discusses the basic concepts of the theory of groups. It begins with the definition of a group and the properties of groups. The next section discusses the subgroups of a group and the quotient groups. The third section discusses the homomorphisms of groups and the isomorphism theorems. The fourth section discusses the direct products of groups and the direct sums of vector spaces. The fifth section discusses the Sylow theorems and the structure of finite groups. The sixth section discusses the representation theory of groups and the character theory. The seventh section discusses the Galois theory of field extensions and the Galois groups of polynomials. The eighth section discusses the theory of solvable groups and the Galois theory of solvable extensions. The ninth section discusses the theory of simple groups and the classification of finite simple groups. The tenth section discusses the theory of Lie algebras and the representation theory of Lie algebras. The eleventh section discusses the theory of algebraic groups and the representation theory of algebraic groups. The twelfth section discusses the theory of p-adic groups and the representation theory of p-adic groups. The thirteenth section discusses the theory of reductive groups and the representation theory of reductive groups. The fourteenth section discusses the theory of adelic groups and the representation theory of adelic groups. The fifteenth section discusses the theory of automorphic forms and the representation theory of automorphic forms. The sixteenth section discusses the theory of modular forms and the representation theory of modular forms. The seventeenth section discusses the theory of theta functions and the representation theory of theta functions. The eighteenth section discusses the theory of elliptic curves and the representation theory of elliptic curves. The nineteenth section discusses the theory of abelian varieties and the representation theory of abelian varieties. The twentieth section discusses the theory of algebraic surfaces and the representation theory of algebraic surfaces. The twenty-first section discusses the theory of algebraic threefolds and the representation theory of algebraic threefolds. The twenty-second section discusses the theory of algebraic fourfolds and the representation theory of algebraic fourfolds. The twenty-third section discusses the theory of algebraic fivefolds and the representation theory of algebraic fivefolds. The twenty-fourth section discusses the theory of algebraic sixfolds and the representation theory of algebraic sixfolds. The twenty-fifth section discusses the theory of algebraic sevenfolds and the representation theory of algebraic sevenfolds. The twenty-sixth section discusses the theory of algebraic eightfolds and the representation theory of algebraic eightfolds. The twenty-seventh section discusses the theory of algebraic ninefolds and the representation theory of algebraic ninefolds. The twenty-eighth section discusses the theory of algebraic tenfolds and the representation theory of algebraic tenfolds. The twenty-ninth section discusses the theory of algebraic elevenfolds and the representation theory of algebraic elevenfolds. The thirtieth section discusses the theory of algebraic twelvefolds and the representation theory of algebraic twelvefolds.

1.1. Groups

A group is a set G with a binary operation \cdot satisfying the following properties:

- Associativity: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in G$.
- Identity: There exists an element $e \in G$ such that $e \cdot a = a \cdot e = a$ for all $a \in G$.
- Inverse: For each $a \in G$, there exists an element $a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = e$.

If H is a subset of G such that H is a group under the operation \cdot , then H is called a subgroup of G . The quotient group G/H is defined as the set of cosets aH for $a \in G$, with the operation $(aH) \cdot (bH) = (a \cdot b)H$.

A homomorphism ϕ from a group G to a group H is a map $\phi: G \rightarrow H$ such that $\phi(a \cdot b) = \phi(a) \cdot \phi(b)$ for all $a, b \in G$. The kernel of ϕ is the set $\ker \phi = \{a \in G \mid \phi(a) = e_H\}$, and the image of ϕ is the set $\text{Im } \phi = \{\phi(a) \in H \mid a \in G\}$.

The First Isomorphism Theorem states that if $\phi: G \rightarrow H$ is a homomorphism, then $\text{Im } \phi \cong G/\ker \phi$. The Second Isomorphism Theorem states that if $\phi: G \rightarrow H$ is a homomorphism and K is a subgroup of G , then $\phi(K) \cong \phi(K)/\phi(\ker \phi)$.

The Third Isomorphism Theorem states that if $\phi: G \rightarrow H$ is a homomorphism and K is a normal subgroup of G , then $\phi(K) \cong \phi(K)/\phi(\ker \phi)$. The Fourth Isomorphism Theorem states that if $\phi: G \rightarrow H$ is a homomorphism and K is a normal subgroup of G , then $\phi(K) \cong \phi(K)/\phi(\ker \phi)$.